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A MATHEMATICAL MODEL
of the
BASIC ELECTRO-HYDRODYNAMIC PROCESS
INCLUDING EFFECTS OF TURBULENCE

by

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ABSTRACT

A mathematical model of the basic electro-hydrodynamic (EHD) process is developed and described. The carrier fluid is treated as incompressible and turbulent. The injected particles are treated as uniform in mass and electrical charge. The analysis is broken down into three phases, namely, the basic flow, the perturbation due to injection of mass, and the perturbation due to introduction of electrical charge. This method greatly simplifies and improves the analysis. A final system of nine basic partial differential equations is obtained. These equations, along with the appropriate boundary conditions, fix the fluid and particle velocities and particle density at all points in the field. The basic equations are developed in a fully non-dimensional form.

The mathematical model here presented is unique in its analytical approach and in its treatment of turbulence effects. Through computer simulation, it offers new possibilities for the study and development of EHD power generation systems.

The analytical model has been developed to the point where it is ready for computer programming. Such a program would be useful for estimating optimum design parameters and performance possibilities for a wide variety of axi-symmetric configurations and a wide range of operating conditions.

Unfortunately the work has now been halted by shortage of funds. Because of the value of this research, it is recommended that this project be resumed and continued into the next stage, which is the stage of actual programming and computation.

CONTENTS

1.0 INTRODUCTION

This report covers the theoretical phase of a combined experimental and theoretical program on the fundamentals of electro-hydrodynamic (EHD) processes, as carried out within the Department of Aeronautics, Naval Postgraduate School during 1970. The work was supported by the Naval Air Systems Command as a continuation of AIRTASK A 340340C/551A/ORO/002010 under the technical cognizance of Dr. H. R. Rosenwasser, Code AIR-310.

The experimental part of the program, carried out by Professors O. Biblarz and K. E. Woehler, is summarized separately in reference (1). Therefore, the present report summarize only the results of the theoretical investigation as carried out by the present writer.

The principal aim of the theoretical work was to devise a suitable mathematical model in terms of which the fundamental characteristics of EHD processes, especially power generation processes, could be numerically simulated, analyzed, understood and optimized.

A subsidiary aim was to include, so far as feasible, the specific effects of fluid turbulence on the behavior of EHD systems.

In the course of the analytical work a series of progressively more complex mathematical models were devised and studied, starting with relatively simple one-dimensional formulations. An important purpose of this phase of the research was to develop analytical approaches which represent a reasonable compromise between physical realism on the one hand and practical computability on the other.

This work has now culminated in the development of a two-dimensional model which is suitable for application to a wide variety of axi-symmetric

configurations. The analysis and derviation of this mathematical model constitutes the subject of the present report.

The model herein described is now essentially complete on its analytical side, although it has not as yet been programmed for actual computation. Such computation would naturally become the objective of the next phase of the research.

In connection with the turbulence problem, it was found that turbulence theory, at its present level of development, does not provide a fully adequate basis for the desired application to EHD. However, it does permit the inclusion of certain significant turbulence effects on a semi-empirical basis. These effects have been incorporated into the EHD model discussed herein.

Because of the limitations of existing turbulence theory, not only in connection with EHD but also quite generally, it was decided to make a collateral research effort to simulate, at least one a two dimensional basis, the detailed mechanism of turbulence itself. This is an ambitious goal, but the ultimate success of this effort should significantly increase our abilities to cope not only with the implications for EHD but also for many other applications as well. Progress in this connection has been gratifying. This collateral research will not be discussed in the present report but is fully documented in references (2) and (3).

The present report presents a mathematical model of the actual EHD process. However, this document is largely in the nature of a progress report on the initial analytical phase of the research, not a completed study. It concentrates therefore on basic concepts and methods, leaving actual computations for the next phase of the work. Its main significance lies in the fact that all the essential features of a rather complex yet computable model are defined and developed herein.

In addition, the various concepts are illustrated by application to the idealized case of an uniform circular duct containing charged particles distributed uniformly over the volume enclosed.

2.0 RECOMMENDATION

The mathematical model of EHD processes described and developed in this report is unique in that it includes effects of mass and momentum transport caused by fluid turbulence. The model is now ready for actual computer programming. Such a program could compute the performance of various axi-symmetric configurations over a wide range of operating conditions. It would provide a valuable and economical method for estimating the possibilities of various designs.

For these reasons it is strongly recommended that this work be continued through the next phase, which is the phase of actual programming and computation.

Unfortunately, at the time of writing, the program is being curtailed because of lack of funds. However, in view of the potential value of this research, it is recommended that the question of funding be reviewed at the earliest opportunity.

3.0 ANALYSIS BY METHOD OF PERTURBATIONS

It proves to be highly advantageous in constructing a mathematical model of the EHD process to proceed by a method of perturbations. Specifically, we start with the basic flow of the carrier fluid in the absence of particles. After the characteristics of the basic flow are established, we consider a first perturbation which consists of the addition of secondary particles to the main flow. At this stage, the secondary particulate flow is treated as electrically neutral, and the initial perturbation therefore accounts only for the mass and momentum effects of the added particles. Finally, a secondary perturbation is introduced which consists solely of the addition of electrical charges to the particles previously considered. The final flow field is therefore described in terms of the basic flow plus the effects added by the above two perturbations.

The separation of the momentum and electrical effects in the form of small perturbations superimposed on a known underlying flow field is important for several reasons. In the first place, the perturbation effects are in themselves quite small, and would tend to be lost in any analysis which simply lumps them in with gross flow field phenomena in an indiscriminate manner. Secondly, the fact that the perturbations are small does permit the corresponding solutions to be linearized. This greatly simplifies the analysis, yet does so without introducing appreciable errors. Thirdly, the perturbation method enables the problem to be sub-divided and solved in successive stages.

We proceed, therefore, in the following sections to consider each of these successive stages in turn.

4.0 COMPRESSIBILITY AND TURBULENCE

In an actual electro-fluid dynamic energy conversion device, the flow would normally be both compressible and turbulent. The question arises, therefore, as to what extent it is necessary and possible to take these specific effects into account in our initial mathematical model.

In general, it can be said that it is simply not possible to devise one comprehensive mathematical model that takes into account all significant physical effects and which is at the same time practically computable. We are obliged, therefore, to work with a number of separate and simplified models, each of which can shed light on certain selected aspects of the overall process. Only in this piece-meal fashion can actual progress be achieved toward mastering the various complexities of the problem.

Thus, even if the carrier fluid is in fact compressible, if the Mach number in the flow region where the EHD process is actually taking place is not too high, a local analysis on the basis of incompressible flow theory should still be reasonably adequate. The incentive for adopting this approach is, of course, that the analysis is thereby greatly simplified. Consequently, the main effort can be directed toward the details of the EHD process which are, after all, the primary objects of interest in our problem. Therefore, in the present discussion, we restrict our model to that of incompressible flow.

Somewhat similar considerations apply to the problem of turbulence. However, the detailed mechanism of turbulence is quite closely related to

detailed EHD phenomena which are our primary concern. Hence there is a very strong incentive for attempting to include turbulence effects in our mathematical model, even at the present early stage of the overall development.

Unfortunately, this is very difficult to do as turbulence theory itself, quite apart from possible applications to EHD problems, is still in a very unsatisfactory state of development. In fact, it can be said that before we can adequately apply turbulence theory to the EHD problem, we shall first have to develop a more adequate theory of turbulence! At first glance, any such effort might seem to be too remote from the immediate objective which aims directly at the EHD problem itself. However, in science, the long way round is sometimes the short way home.

We have therefore adopted a two pronged strategy. On the one hand, a somewhat simplified treatment of turbulence is incorporated into the present model of the EHD process as discussed below. On the other hand, a collateral effort has been launched on the basic turbulence problem itself. This latter phase is discussed fully in references (2) and (3).

It suffices here to remark simply that the collateral study has made gratifying progress. If this effort is continued, it should make possible a far more adequate treatment of turbulence in the specific application to EHD.

5.0 THE BASIC FLOW

In the great majority of cases, the basic flow in the EHD portion of an EHD energy converter will be axi-symmetric. Accordingly, we develop here the detailed equations, in cylindrical coordinates, for an arbitrary

axi-symmetric configuration. One specific case which exhibits these relations to advantage in a particularly simple form is that of flow thru a uniform duct of circular cross-section. The general equations are therefore applied to this particular case and simplified accordingly. This example serves as an instructive model for applying the theory here developed to other more complex configurations.

In dealing with turbulent flow there are, in principle, two basic alternatives at our disposal. In the first place, we can, in theory, deal with the actual fluctuating flow field, in all its true complexity. The theoretical advantage is that the equations of continuity and motion, along with the appropriate boundary conditions, suffice in principle to determine all features of the flow in complete detail. Actually, however, this alternative is not practical. The fluctuations possess a structure of immense complexity. They encompass three spaces dimensions and one time dimension. Each of these four dimensions involves a spectrum which spans a great band width, so that the entire system involves myriads of degrees of freedom. Moreover, the equations are non-linear, which has the effect of coupling all these degrees of freedom.

The second alternative at our disposal is to average the equations of continuity and motion in an effort to confine attention to the mean flow itself. In this case, it is found that the only effect of the fluc-uations on the mean flow is through the so-called Reynolds stresses. While this method goes far toward eliminating the complexities of the first approach, it suffers from the defect that the process of averaging the basic equations leads to the loss of certain essential information. This shows up mathematically through the fact that the system of equation thereby obtained is indeterminate. As it turns out, these equations

define exactly how the Reynolds stresses affect the mean flow, but fail completely to provide any clue concerning the reciprocal effect of the mean flow on the Reynolds stresses. Consequently, in order to establish a determinate system of equations, it becomes mandatory to introduce additional hypotheses which purport to define the Reynolds stresses as functions of the mean flow field. This is the closure problem of turbulence. Unfortunately, as much research has shown, this in itself is an extraordinarily difficult problem, and while empirical approximations have been developed for various specific configurations, no general solution to this problem has as yet been fully established. Much current research is being directed at this objective, however, and considerable progress is now being made. Reference (6) provides an excellent review and summary of this work. Moreover, even if there were an acceptable solution to the closure problem of turbulence now available, that in itself would not be adequate for the application to EHD. Clearly, we require for the latter purpose, not merely a knowledge of the Reynolds stresses themselves, but in addition, considerable information about turbulent diffusion, which depends in turn on the fine details of the turbulent motion. Hence the needs of the EHD problem provide part of the justification for attempts to construct a model which reveals these details. An encouraging degree of success has been achieved in this connection by means of computer simulation of two dimensional turbulence, as described in references (2) and (3).

The most common hypothesis employed for providing a plausible closure is to relate the Reynolds stresses to the mean flow by means of a so-called eddy viscosity. This amounts to treating the Reynolds stresses much as if they were simply additional viscous stresses, attributable to an additional viscosity of some kind. In fact, it then becomes convenient simply

to lump the viscous and Reynolds stresses together into a common term, and to attribute this to the action of an overall effective or eddy viscosity ε. There are ample experimental and theoretical grounds for asserting that such a simple eddy viscosity concept cannot be entirely correct. Nevertheless, it does suffice to yield an acceptable approximation to the mean flow provided that the effective viscosity ε be prescribed judiciously. It turns out that ε is not a fixed property of the fluid, but rather a property of the flow, and that it can vary considerably over the field. Empirical rules are available for estimating ε for various flow configurations, but the detailed enunciation of these goes beyond the scope of the present discussion. Note, however, that use of the eddy viscosity concept, in effect plays down any direct dealing with the actual turbulent fluctuations themselves, and concentrates instead primarily upon the mean flow. Moreover, the mean flow is treated largely as if it were an actual smooth laminar flow rather than being merely the hypothetical mean of a strongly fluctuating flow field.

Consequently, the introduction of eddy viscosity into our overall model means that we are taking into account the effects of turbulence mainly insofar as these effects are reflected in the form of the mean velocity distribution. Since the EHD process is certainly affected by the mean velocity distribution, this procedure does have a certain value. On the other hand, some of the important EHD effects of the detailed turbulent motion, effects which determine such a vital factor as the potential gradient at which electrical breakdown occurs, cannot be fully represented in this way. To a certain extent, some of these aspects can be explored by means of separate supplementary models, but we shall not digress at this point to consider these additional possibilities. Fortunately, however, there is one other significant effect of turbulence which we

have been able to incorporate into the present model in a somewhat simplified but reasonably satisfactory form, and that is the effect of turbulent diffusion, as explained in a later section.

In the following analysis the mean flow is always taken as steady. The turbulent fluctuations are, of course, unsteady, but all mean properties of the turbulence such as the Reynolds stresses, and so forth, remain steady.

6.0 NON-DIMENSIONALIZATION OF EQUATIONS: PARAMETRIC STUDIES

Any quantity which is denoted in this report by an ordinary letter symbol, say $\overline{X}$ for example, shall be understood to be expressed in dimensionless units, as explained below. However, if it is written in the form $\hat{X}$, the mark ^ shall specifically designate that the quantity is expressed in conventional dimensional units.

All quantities which occur in the present problem of EHD have dimensions which are reducible to the fundamental dimensions of force, length, time, and electrical charge, or to any equivalent set of four basic dimensions. In turn this means that all quantities can be non-dimensionalized in terms of four suitably chosen fundamental parameters of the problem. It is suggested that the most convenient and physically significant set to choose for this purpose are the following four quantities:

$\hat{\rho}$ = density of carrier fluid.

$\hat{\ell}$ = a characteristic linear dimension. For a uniform round duct the radius $\hat{R}$ may serve as the characteristic length.

$\hat{V}$ = a characteristic velocity. For flow through a uniform round duct, the mean velocity $\hat{\overline{U}}$ may serve as the characteristic velocity.

$\hat{\Delta\phi}$ = a characteristic potential difference. The magnitude of the overall potential difference $|\hat{\phi}_1 - \hat{\phi}_0|$ between inlet and outlet may serve as the characteristic potential difference.

It now follows from well known principles of dimensional analysis that if $\hat{\underline{X}}$ be any quantity whatever which occurs in the problem, then a dimensionless version of $\hat{\underline{X}}$, call it $\bar{\underline{X}}$, can always be obtained in the form

$$\bar{\underline{X}} = \frac{\hat{\underline{X}}}{\hat{\rho}^a \hat{\ell}^b \hat{V}^c \hat{\Delta\phi}^d} \tag{6-1}$$

The exponents a, b, c, d can always be found such that the denominator of (6-1) will have the same dimensions as the numerator, thus guaranteeing the non-dimensionality of $\bar{\underline{X}}$.

An important corollary of this is the following. Consider any equation in the present analysis. Let the various quantities occurring in this equation be denoted by symbols $\hat{\underline{X}}_1$, $\hat{\underline{X}}_2$, $\hat{\underline{X}}_3$.... Let the dimensionless counterparts of these be denoted by $\bar{\underline{X}}_1$, $\bar{\underline{X}}_2$, $\bar{\underline{X}}_3$.... Now if the original quantities $\hat{\underline{X}}_1$, $\hat{\underline{X}}_2$, $\hat{\underline{X}}_3$... in the equation all be replaced by their respective dimensionless counterparts $\bar{\underline{X}}_1$, $\bar{\underline{X}}_2$, $\bar{\underline{X}}_3$..., the equation will still remain equally valid, consistent and applicable! This amounts to saying that all variables, all parameters, and all equations of the present analysis can be consistently non-dimensionalized on the basis of the four reference parameters $\hat{\rho}$, $\hat{\ell}$, $\hat{V}$, and $\hat{\Delta\phi}$.

The foregoing type of non-dimensionalization has been carried out consistently throughout this report. Consequently, the resulting equations will be found to contain a small number of characteristic dimensionless parameters. These are then the really fundamental parameters of the problem. One of these, for example, turns out to be the familiar Reynold's number

$\left(\frac{\hat{V}\hat{\ell}}{\hat{\nu}}\right)$. Other less familar but equally significant physical parameters are also evident in the equations, some of them linking the electrical aspects of the problem to the hydrodynamic.

The characteristic dimensionless parameters which have been obtained in this way, when utilized in conjunction with a working computer program, can provide a rational basis for an orderly parametric study of the problem. Such parametric studies are the means whereby the optinum range of design parameters may be found and peak performance potential investigated.

7.0 SYMBOLS

All plain symbols denote non-dimensionalized quantities in the sense explained in section 6. Addition of the mark ˆ over a symbol indicates that the symbol now denotes the dimensional form of the quantity, with conventional units. Where conventional units are listed below, they apply only to the dimensional form (ˆ) of the quantity.

a, b, c, d = constants

D = diffusion coefficient

E = turbulent energy per unit mass

$\vec{e}_x, \vec{e}_r, \vec{e}_\theta$ = unit vectors in axial, radial and circumferential directions, respectively

$\vec{f}$ = net Reynolds plus viscous force per unit mass on a fluid element

f_x, f_n, f_θ = components of $\vec{f}$ ($f_\theta = 0$)

$\vec{f}_p$ = net pressure force (including turbulence pressure) per unit volume on a fluid element

$f_{xp}, f_{rp}, f_{\theta p}$ = components of $\vec{f}_p$ ($f_{\theta p} = 0$)

$\vec{F}_v{}'$ = viscous drag force on a single particle for primary perturbation

$\vec{F}_v{}''$ = viscous drag force on a single particle for secondary perturbation

$\vec{F}_E{}''$ = electrical force on a single particle

$\vec{f}_v{}'$ = viscous reaction of particles on carrier fluid, per unit mass of carrier fluid, for primary perturbation

$\vec{f}_v{}''$ = viscous reaction of particles on carrier fluid, per unit mass of carrier fluid, for secondary perturbation

$\vec{f}_E{}''$ = electrical force on particles per unit mass of carrier fluid

$\vec{J}$ = current flux density

$\hat{\ell}$ = a characteristic length of the configuration (m.)

ℓ = distance from inlet to outlet stations

m_p = mass of a single particle

m' = local mass density of particles for primary perturbation

m'' = local mass density of particles for secondary perturbation

$\dot{m}$ = mass injection rate of particles expressed as mass per unit volume or mass per unit area, as appropriate

n' = number density of particles for primary perturbation

n'' = number density of particles for secondary perturbation

P_t = turbulence pressure as defined by Eq. (9-7)

P_f = ordinary fluid pressure

P = total pressure = sum of ordinary fluid pressure plus turbulence pressure

p' = primary perturbation pressure

p'' = secondary perturbation pressure

q_p = electrical charge on a single particle

R_p = particle radius

$\vec{U}$ = velocity of basic flow

U_x, U_r, U_θ = components of U ($U_\theta = 0$)

u' = primary perturbation velocity of carrier fluid

u'_x, u'_r, u'_θ = components of u' ($u'_\theta = 0$)

$\vec{u}''$ = secondary perturbation velocity of carrier fluid

u''_x, u''_r, u''_θ = components of u'' ($u''_\theta = 0$)

U' = first derivative of velocity for flow in a uniform duct

U'' = second derivative of velocity for flow in a uniform duct

U'_w = slope of velocity curve at wall

$\vec{v}'$ = particle velocity relative to carrier fluid for primary perturbation

v'_x, v'_r, v'_θ = components of v' ($v'_\theta = 0$)

$\vec{v}''$ = particle velocity relative to carrier fluid for secondary perturbation

v''_x, v''_r, v''_θ = components of $\vec{v}''$ ($v''_\theta = 0$)

v_p = volume of a single particle

u, v, w = components of turbulent velocity fluctuation

$\hat{V}$ = a characteristic velocity of the system (m/sec)

v^* = friction velocity

$\overline{v}_t$ = mean mixing velocity in turbulent diffusion

$\vec{w}'$ = mean effective diffusion velocity for primary perturbation

$\vec{w}''$ = mean effective diffusion velocity for secondary perturbation

x, r, θ = axial, radial, and angular coordinates, respectively, in cylindrical coordinate system

$\underline{\overline{X}}_1$, $\underline{\overline{X}}_2$, $\underline{\overline{X}}_3$ = arbitrary generalized variables

γ = ratio of turbulent mass diffusion coefficient to turbulent eddy viscosity

δ = dimensionless constant defined in Eq. (10-23)

ε = eddy viscosity plus molecular viscosity

ε_o = relative dielectric constant of field

ζ' = particle stream function for primary perturbation as defined by Eq. (11-14)

ζ'' = particle stream function for secondary perturbation

θ' = particle potential function for primary perturbation as defined by Eq. (11-14)

K = 0.36 = empirical dimensionless constant in von Karman's mixing length theory

λ = mixing length

ν = kinematic viscosity

$\hat{\rho}$ = density of carrier fluid (Kg/m^3)

ρ''_e = charge density

ρ_p = density of an individual particle

$\left.\begin{matrix}\tau_{xx}, \tau_{rr}, \tau_{\theta\theta}, \\ \tau_{r\theta}, \tau_{\theta x}, \tau_{xr}\end{matrix}\right\}$ = stress components representing sum of viscous plus Reynolds stresses in carrier fluid

τ_w = shear stress at wall

Φ'' = electric potential

$\Delta\hat{\phi}$ = a characteristic electric potential difference of the configuration (volts)

ϕ_0 = electric potential at inlet station (volts)

ϕ_1 = electric potential at exit station (volts)

Ψ = stream function of basic flow

ψ' = stream function of primary perturbation

ψ'' = stream function of secondary perturbation

$\vec{\Omega}$ = vorticity of basic flow

$\vec{\omega}'$ = vorticity of primary perturbation

$\vec{\omega}''$ = vorticity of secondary perturbation

8.0 THE CONTINUITY EQUATION OF THE BASIC FLOW

Invoking the principle of the conservation of mass leads to the following equation which must be satisfied. In vector notation

$$\nabla.\vec{U} = 0 \tag{8-1}$$

One optional way to satisfy Eq. (4-1) is to define the velocity U in terms of a stream function Ψ. For axi-symmetrical flows this relation takes the form

$$\vec{U} = \frac{1}{r}\nabla\Psi \times \vec{e}_\theta \tag{8-2}$$

Direct substitution of Eq. (8-2) into (8-1) will verify that continuity requirements are satisfied identically.

The foregoing relations (8-1) and (8-2) can also be written in scalar form. Using cylindrical coordinates, x, r, θ gives (4-1) in the form

$$\frac{\partial U_x}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left(rU_r\right) = 0 \tag{8-3}$$

Eq. (8-2) then becomes

$$U_x = \frac{1}{r}\frac{\partial\Psi}{\partial r}$$

$$U_r = -\frac{1}{r}\frac{\partial\Psi}{\partial x} \tag{8-4}$$

Thus, in solving for initially unknown flows, the basic relations may be expressed in terms of the stream function ψ rather than in terms of the actual velocity components U_x and U_r. This procedure assures us that the solution subsequently attained will automatically satisfy the continuity requirement.

However, in some problems the basic flow field may be presumed to be known, either by experiment or by prior analysis. In such cases it is often clearer to express key relations directly in terms of actual velocities U_x and U_r rather than in terms of stream function Ψ. This is the course we will actually follow in the subsequent discussion with the understanding that the stream function/velocity relation (8-2) or (8-4) can always be introduced when and if it turns out to be needed.

9.0 VISCOUS AND REYNOLDS STRESSES AND FORCES

We choose to lump the viscous and Reynolds stresses together and express them both in terms of an effective overall eddy viscosity ε as discussed earlier. Also, all quantities are expressed in terms of non-dimensionalized variables, as previously explained. The six dimensionless stresses become

$$
\begin{aligned}
\tau_{xx} &= 2\varepsilon \left(\frac{\partial U_x}{\partial x}\right) & \qquad \tau_{r\theta} &= 0 \\
\tau_{rr} &= 2\varepsilon \left(\frac{\partial U_r}{\partial r}\right) & \qquad \tau_{\theta x} &= 0 \\
\tau_{\theta\theta} &= 2\varepsilon \left(\frac{U_r}{r}\right) & \qquad \tau_{xr} &= \varepsilon \left(\frac{\partial U_r}{\partial x} + \frac{\partial U_x}{\partial r}\right)
\end{aligned}
\tag{9-1}
$$

Actually the above expressions include only the deviatoric part of the stress tensor. The isotropic part of the tensor is more conveniently lumped with the pressure term as explained later.

The above stress system produces a net unbalanced force on each infinitesimal element of the fluid. The corresponding components of force per unit mass are simply

$$
\begin{aligned}
f_x &= \left(\frac{\partial \tau_{xx}}{\partial x}\right) + \frac{1}{r}\frac{2}{\partial r}\left(r\tau_{xr}\right) \\
f_r &= \left(\frac{\partial \tau_{xr}}{\partial x}\right) + \frac{1}{r}\frac{2}{\partial r}\left(r\tau_{rr}\right) - \frac{\tau_{\theta\theta}}{r} \\
f_\theta &= 0
\end{aligned}
\tag{9-2}
$$

Eqs. (9-1) may now be substituted into the force expressions (9-2) The result is simply

$$f_x = \frac{\partial}{\partial x}\left[2\varepsilon\left(\frac{\partial U_x}{\partial x}\right)\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[r\varepsilon\left(\frac{\partial U_r}{\partial x} + \frac{\partial U_x}{\partial r}\right)\right]$$

$$f_r = \frac{\partial}{\partial x}\left[\varepsilon\left(\frac{\partial U_r}{\partial x} + \frac{\partial U_x}{\partial r}\right)\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[r2\varepsilon\left(\frac{\partial U_r}{\partial r}\right)\right] - 2\varepsilon\left(\frac{U_r}{r^2}\right) \qquad (9\text{-}3)$$

$$f_\theta = 0$$

In situations where the distribution of the eddy viscosity ε is an already known function, it is most convenient to express the forces directly in the form given in (9-3). However, if we are in the process of solving for the distribution of ε, it becomes advantageous to expand Eqs. (9-3) as follows.

$$f_x = \varepsilon\left\{2\left(\frac{\partial^2 U_x}{\partial x^2}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial U_r}{\partial x} + \frac{\partial U_x}{\partial r}\right)\right]\right\}$$

$$+ 2\left(\frac{\partial\varepsilon}{\partial x}\right)\left(\frac{\partial U_x}{\partial x}\right) + \left(\frac{\partial\varepsilon}{\partial r}\right)\left(\frac{\partial U_r}{\partial x} + \frac{\partial U_x}{\partial r}\right)$$

$$(9\text{-}4)$$

$$f_r = \varepsilon\left\{\frac{\partial}{\partial x}\left[\left(\frac{\partial U_r}{\partial x} + \frac{\partial U_x}{\partial r}\right)\right] + \frac{2}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial U_r}{\partial r}\right)\right] - \frac{2U_r}{r^2}\right\}$$

$$+ \left(\frac{\partial\varepsilon}{\partial x}\right)\left(\frac{\partial U_r}{\partial x} + \frac{\partial U_x}{\partial r}\right) + 2\left(\frac{\partial\varepsilon}{\partial r}\right)\left(\frac{\partial U_r}{\partial r}\right)$$

$$f_\theta = 0$$

Of course the Reynolds stresses also contribute, in addition to the six important deviatoric stress components, an isotropic pressure - like term related to the kinetic energy of turbulence. Specifically the dimensionless turbulent energy per unit mass is

$$E = \frac{1}{2}\left(\bar{u}_x^2 + \bar{u}_r^2 + \bar{u}_\theta^2\right) \qquad (9\text{-}5)$$

The corresponding dimensionless turbulent pressure is given by

$$P_t = \frac{1}{3}\left(\bar{u}_x^2 + \bar{u}_r^2 + \bar{u}_\theta^2\right) \qquad (9\text{-}6)$$

from which it follows that

$$P_t = \frac{2}{3}E \tag{9-7}$$

Naturally, gradients in P_t also create small net unbalanced forces on the fluid elements which should not be overlooked. However, by far the most convenient way to include this factor in the analysis is simply to lump the turbulence pressure P_t in with the ordinary fluid pressure P_t. We then denote the sum by the single symbol

$$P = P_f + P_t \tag{9-8}$$

The three dimensionless net pressure forces per unit volume then become simply

$$\begin{aligned} \left(f_x\right)_p &= -\left(\frac{\partial P}{\partial x}\right) \\ \left(f_n\right)_p &= -\left(\frac{\partial P}{\partial r}\right) \\ \left(f_\theta\right)_p &= 0 \end{aligned} \tag{9-9}$$

The foregoing analysis indicates that all of the viscous, turbulent and pressure forces can be expressed in terms of the four dimensionless functions U_x, U_r, ε, and P.

If the particular flow configuration we are considering happens to be one which is known to possess a similarity solution, then of course it becomes highly advantageous to change from cylindrical coordinates to the appropriate similarity coordinates. In that case Eqs. (9-4) and all other key equations must be transformed to the new coordinates.

Now consider the special case of steady turbulent flow in a uniform round duct. The dimensionless stresses (9-1) reduce to

$$\tau_{xx} = 0 \qquad \tau_{r\theta} = 0$$

$$\tau_{rr} = 0 \qquad \tau_{\theta x} = 0 \tag{9-10}$$

$$\tau_{\theta\theta} = 0 \qquad \tau_{xr} = \varepsilon\left(\frac{\partial U_x}{\partial r}\right)$$

The corresponding net forces (9-3) now become simply

$$f_x = \frac{1}{r}\frac{\partial}{\partial r}\left[r\varepsilon\left(\frac{\partial U_x}{\partial r}\right)\right]$$

$$f_r = 0 \tag{9-11}$$

$$f_\theta = 0$$

The net Reynolds force $\vec{f}$, whose components are given by Eqs. (9-3) or (9-4) can also be expressed in vector notation as follows

$$\vec{f} = \nabla\cdot\left(\varepsilon\nabla\vec{U}\right) \tag{9-12}$$

This form is seen to have the advantage of being extremely concise, and it is often useful for just this reason. On the other hand, the scalar form (9-3) has the merit of showing exactly what the detailed calculation of $\vec{f}$ actually entails.

Similarly, the net pressure force whose components are given by Eqs. (9-9) can also be reduced to the simple vector expression

$$\vec{f}_p = -\nabla P \tag{9-13}$$

10.0 THE EQUATIONS OF MOTION AND THEIR SOLUTION

By making use of the foregoing expressions for the forces, the equations of motion of the basic steady mean flow may be written in terms of dimensionless variables in vector form

$$\begin{aligned}\vec{U}\cdot\nabla\vec{U} &= -\nabla P + \vec{f}\\ &= -\nabla P + \nabla\cdot\left(\varepsilon\nabla\vec{U}\right)\end{aligned} \tag{10-1}$$

By applying the operators $\nabla\times$ and $\nabla\cdot$ to this equation, two related equations of fundamental importance can be obtained. First we define the vorticity $\vec{\Omega}$ as follows:

$$\vec{\Omega} = \nabla\times\vec{U} \tag{10-2}$$

Taking the curl of Eq. (10-1) gives the basic vorticity equation

$$\vec{U}\cdot\nabla\vec{\Omega} = 0 + \nabla\times\left[\nabla\cdot\left(\varepsilon\nabla\vec{U}\right)\right] \tag{10-3}$$

Notice that application of the curl operator $\nabla\times$ has resulted in the elimination of the unknown pressure term from (10-3). Hence this fundamental vorticity equation contains only the dimensionless functions ε and $\vec{U}$.

In order to solve Eq. (10-3) an auxiliary hypothesis must be made concerning the dependence of ε on the $\vec{U}$ distribution. Let us denote this hypothesis by the symbolic statement

$$\varepsilon = \varepsilon(\vec{U}) \tag{10-4}$$

Eq. (10-3) and hypothesis (10-4), along with the appropriate boundary conditions, then fix the detailed solution for ε and $\vec{U}$.

In some instances the function $\vec{U}$ is known by experiment or prior analysis. In such cases, hypothesis (10-4) is no longer needed, and the distribution of ε over the field may be found directly from Eq. (10-3).

Next consider the application of the divergence operator to (10-3). The results of this operation may be rearranged into the form

$$\nabla^2 P = -\nabla\cdot\left(\vec{U}\cdot\nabla\vec{U}\right) - \nabla\cdot\vec{f} \tag{10-5}$$

Once U and ε have been found from Eqs. (10-3) and (10-4), then Eq. (10-5) may be used to solve for the corresponding pressure distribution.

It is useful to list the scalar equations which correspond to the foregoing vector relations. Thus Eq. (10-1) gives

$$U_x\left(\frac{\partial U_x}{\partial x}\right) + U_r\left(\frac{\partial U_r}{\partial r}\right) = -\frac{\partial P}{\partial x} + f_x$$

$$U_x\left(\frac{\partial U_r}{\partial x}\right) + U_r\left(\frac{\partial U_r}{\partial r}\right) = -\frac{\partial P}{\partial r} + f_r \tag{10-6}$$

Eq. (10-2) gives

$$\Omega = \left(\frac{\partial U_r}{\partial x} - \frac{\partial U_x}{\partial r}\right) = -\frac{1}{r}\left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial r^2}\right) \tag{10-7}$$

We write (10-3) in the form

$$U_x\left(\frac{\partial \Omega}{\partial x}\right) + U_r\left(\frac{\partial \Omega}{\partial r}\right) = \left(\frac{\partial f_r}{\partial x} - \frac{\partial fx}{\partial r}\right) \tag{10-8}$$

where the f's are as defined by Eqs. (9-3) or (9-4).

Similarly, Eq. (10-5) may be expanded to the form

$$\frac{\partial^2 P}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right)$$

$$= -\frac{\partial}{\partial x}\left\{U_x\left(\frac{\partial U_x}{\partial x}\right) + U_r\left(\frac{\partial U_x}{\partial r}\right)\right\} - \frac{1}{r}\frac{\partial}{\partial r}\left\{r\left[U_x\left(\frac{\partial U_r}{\partial x}\right) + U_r\left(\frac{\partial U_r}{\partial r}\right)\right]\right\} \tag{10-9}$$

$$+ \left[\frac{\partial f_x}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left(rf_r\right)\right]$$

No attempt is made here to substitute the f's into Eqs. (10-8) and (10-9) from their definitions (9-3) or (9-4) as this would lead to expressions which are unduly long and cumbersome.

The analysis shows that whereas the equations of motion are defined by (10-1), their actual solution in general will require the prior solution of the more complex and basic forms denoted by (10-3) and (10-4), respectively.

Fortunately, for the case of steady flow through a uniform circular duct of radius R = 1, matters simplify considerably. Specifically, Eqs. (10-6) reduce to the simple forms

$$0 = -\left(\frac{\partial P}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left[r\varepsilon\left(\frac{\partial U_x}{\partial r}\right)\right]$$

$$0 = -\left(\frac{\partial P}{\partial r}\right) + 0 \tag{10-10}$$

Moreover, differentiating the first of these expressions with respect to x, and noting that ε and U are functions of r only, gives

$$\left(\frac{\partial^2 P}{\partial x^2}\right) = 0 \tag{10-11}$$

From (10-11) and the second of Eqs. (10-10) we conclude that

$$\begin{aligned} \left(\frac{\partial P}{\partial x}\right) &= \text{constant} \\ \left(\frac{\partial P}{\partial r}\right) &= 0 \end{aligned} \tag{10-12}$$

Consequently, the first of Eqs. (10-10) can now be partially integrated in the form

$$-\varepsilon\left(\frac{\partial U_x}{\partial r}\right) = \left[-\left(\frac{\partial P}{\partial x}\right)\right]\frac{r}{2} \tag{10-13}$$

In order to complete the integration, it is necessary to introduce an appropriate hypothesis regarding the relation between ε and U_x. For definiteness, we employ the well known mixing length theory of von Karman. In the present application, this theory predicts an eddy viscosity which is given by the expression

$$\varepsilon = K^2 \frac{\left|\frac{\partial U_x}{\partial r}\right|^3}{\left(\frac{\partial^2 U_x}{\partial r^2}\right)^2} \tag{10-14}$$

where K = 0.36, an empirical dimensionless constant. See reference 5, page 512.

With the help of hypothesis (10-14), Eq. (10-13) can be definitely solved. For this purpose we use the notation

$$\frac{\partial U_x}{\partial r} = U' \qquad \frac{\partial^2 U_x}{\partial r^2} = U'' = \left(\frac{\partial U'}{\partial r}\right) \tag{10-15}$$

Also it is useful to introduce in dimensionless form the so-called friction velocity v* which is defined in terms of the dimensionless wall friction as follows

$$v^{*2} = \tau_w = \nu U'_w \tag{10-16}$$

Now note that at the wall itself where r = 1, Eq. (10-13) reduces to the simple result

$$-\nu U'_w = -\frac{\partial P}{\partial x}\frac{1}{2} = v^{*2} \tag{10-17}$$

With the aid of (10-14) and (10-17), Eq. (10-13) may now be rewritten

$$-K^2 \frac{|U'|^3 U'}{(U'')^2} = v^{*2}\, r \tag{10-18}$$

Both sides of (10-18) are positive quantities. We may extract the square root to obtain

$$K\frac{U'^2}{\left(\frac{\partial U'}{\partial r}\right)} = -v^*\sqrt{r} \tag{10-19}$$

The variables are separable. We may therefore integrate from a general radius r to the wall radius r = 1. Thus

$$\int_{U'}^{U'_w} \frac{dU'}{U'^2} = -\frac{K}{v^*}\int_r^1 \frac{dr}{r} \tag{10-20}$$

or

$$-\frac{1}{U'_w} + \frac{1}{U'} = -\frac{2K}{v^*}\left[1-\sqrt{r}\right] \tag{10-21}$$

It is useful to eliminate U'_w from (10-21) by means of (10-17). We may thereby simplify (10-21) to the form

$$\frac{v^*}{U'} = v^*\left(\frac{dr}{dU_x}\right) = -2K\{(1+\delta) - \sqrt{r}\} \tag{10-22}$$

where, for convenience, we have introduced the abbreviation

$$\delta = \frac{\nu}{2Kv^*} \tag{10-23}$$

The second integration is now straight forward. Thus

$$\frac{1}{v^*}\int_U^o dU_x = -\frac{1}{2K}\int_r^1 \frac{dr}{(1+\delta-\sqrt{r})} \tag{10-24}$$

the solution of which gives

$$\left(\frac{U_x}{v^*}\right) = \frac{1}{K}\left\{(1+\delta)\ln\left(\frac{1+\delta-\sqrt{r}}{\delta}\right) - (1-\sqrt{r})\right\} \tag{10-25}$$

Eq. (10-25) defines the final mean velocity distribution for the uniform circular duct in terms of the fuction velocity v* and the dimensionless parameter δ as defined in (10-23). Further integration of (10-25) is possible to establish an additional relationship with the overall volumetric flow rate and mean velocity. However, the analysis has been carried far enough to define all essential features of the flow and this additional integration will not be pursued here.

In this section a complete approach has been defined for analyzing the basic flow field for any axi-symmetric configuration, and the method has been illustrated by application to the particularly simple and significant case of steady flow through a uniform duct of circular cross-section.

11.0 CONTINUITY EQUATION FOR PRIMARY PERTURBATION

Recall that the primary perturbation has been defined as that which results from the addition of mass particles to the basic flow, the mass particles being treated as uncharged at this stage of the analysis. Let $\vec{u}'$ denote the primary perturbation velocity of the carrier fluid. Let $\vec{v}'$ be the relative velocity of the particles with respect to the ambient carrier fluid.

The continuity equation for the carrier fluid perturbation becomes simply

$$\nabla \cdot \vec{u}' = 0 \tag{11-1}$$

Clearly this can be satisfied in the usual way through the introduction of a stream function ψ' such that

$$\vec{u}' = \frac{1}{r} \nabla \psi' \times \vec{e}_\theta \tag{11-2}$$

In scalar terms, the foregoing relations become, respectively,

$$\frac{\partial u'_x}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (r u'_r) = 0 \tag{11-3}$$

and

$$\begin{aligned} u'_x &= \frac{1}{r} \left(\frac{\partial \psi'}{\partial r} \right) \\ u'_r &= - \frac{1}{r} \left(\frac{\partial \psi'}{\partial x} \right) \end{aligned} \tag{11-4}$$

A somewhat different approach is required in applying the continuity equation to the particles themselves, for two reasons.

In the first place, the particles are released, sprayed, entrained, condensed, injected or otherwise introduced into the carrier fluid over certain regions of the flow, and possibly collected, evaporated, separated

or otherwise eliminated over other regions of the flow. From the mathematical view point this can be treated as if it involved the production or annihilation of mass. It is convenient to treat such hypothetical production/annihilation as being distributed over certain volumetric regions, in which case it can be introduced simply into the continuity equation.

In the second place, the mean effective density of the particles may vary strongly over the field.

From the theoretical standpoint the distribution of the particles can be conveniently expressed in terms of $\hat{n}'$, the number density, or number of particles per unit volume, regarded as a continuous variable. (The prime mark is merely a reminder that we are considering the primary perturbation; later the secondary perturbation will be designated by a double prime.)

While the variable $\hat{n}'$ is conceptually simple, its actual numerical values tend to be inconveniently large. Hence we prefer to express the particle distribution in terms of the alternative variable

$$m' = \hat{m}' = \frac{\hat{m}_p \hat{n}'}{\hat{\rho}} \tag{11-5}$$

Notice that m' denotes the ratio of the mean effective density of the particles to that of the carrier fluid. It is therefore a dimensionless variable. We term it the local density ratio. The present analysis is restricted to the case where

$$m' << 1 \tag{11-6}$$

although the generalization to any value of m' would not be difficult.

It is advantageous to develop the subsequent argument first from a vector viewpoint. We let

$\vec{U}$ = basic flow velocity vector

$\vec{u}'$ = carrier fluid perturbation vector

$\vec{v}'$ = relative velocity vector of particles with respect to carrier fluid

m' = local density ratio of particles

m_p = mass of a single particle, taken as equal for all particles

$\dot{m}$ = particle mass generation rate (mass of particles per unit volume per unit time)

The continuity equation for the particles now assumes the form

$$\nabla \cdot [m'(\vec{U} + \vec{u}' + \vec{v}')] = \dot{m} \tag{11-7}$$

Expanding gives

$$m'[\nabla \cdot \vec{U} + \nabla \cdot \vec{u}' + \nabla \cdot \vec{v}'] + \nabla m' \cdot (\vec{U} + \vec{u}' + \vec{v}') = \dot{m} \tag{11-8}$$

However, in this equation, because the components U and u' both separately satisfy continuity, we have

$$\nabla \cdot \vec{U} = \nabla \cdot \vec{u}' = 0 \tag{11-9}$$

Moreover, since the perturbations are definitely small, we have

$$\frac{|\vec{u}'|}{|\vec{U}|} < < 1$$

$$\frac{|\vec{v}'|}{|\vec{U}|} < < 1 \tag{11-10}$$

Consequently, with hardly any error, Eq. (11-8) may be simplified to the form

$$m' \nabla \cdot v' + \nabla m' \vec{U} = \dot{m} \tag{11-11}$$

whereupon finally

$$\nabla\cdot\vec{v}\,' = -\vec{U}\cdot\left(\frac{\nabla m'}{m'}\right) + \left(\frac{\dot{m}}{m'}\right) \tag{11-12}$$

In scalar motation this reduces to the result

$$\left(\frac{\partial v'_x}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}(rv'_r) = -\frac{1}{m'}\left[U_x\left(\frac{\partial m'}{\partial x}\right) + U_r\left(\frac{\partial m'}{\partial r}\right)\right] + \frac{\dot{m}}{m'} \tag{11-13}$$

Eqs. (11-12) or (11-13) show quite clearly that the relative velocity vector $\vec{v}\,'$ <u>cannot</u> be expressed in the ordinary way in terms of a single stream function alone.

It can, however, be described in terms of two basic functions, say θ' and ζ', in the following way. Let

$$\vec{v}\,' = \nabla\theta' + \frac{1}{r}\nabla\zeta' \times \vec{e}_\theta \tag{11-14}$$

where θ' and ζ' are functions only of x and r for the axi-symmetric case. Then the divergence and curl of $\vec{v}\,'$ become respectively,

$$\nabla\cdot\vec{v}\,' = \nabla^2\theta' + 0 \tag{11-15}$$

and

$$\nabla\times\vec{v} = 0 - \vec{e}_\theta\nabla\cdot\left(\frac{\nabla\zeta'}{r}\right) = -\vec{e}_\theta\frac{1}{r}\left(\frac{\partial^2\zeta'}{\partial x^2} + \frac{\partial^2\zeta'}{\partial r^2}\right) \tag{11-16}$$

These relations may now be summarized in scalar terms in the following way

$$\begin{aligned} v'_x &= \frac{\partial\theta'}{\partial x} + \frac{1}{r}\left(\frac{\partial\zeta'}{\partial r}\right) \\ v'_r &= \frac{\partial\theta'}{\partial r} - \frac{1}{r}\left(\frac{\partial\zeta'}{\partial x}\right) \end{aligned} \tag{11-17}$$

$$\begin{aligned} \nabla\cdot\vec{v}\,' = \nabla^2\theta' &= \frac{\partial^2\theta'}{\partial x^2} + \frac{\partial^2\theta'}{\partial r^2} + \frac{1}{r}\frac{\partial\theta'}{\partial r} \\ &= -\frac{1}{m'}\left[U_x\left(\frac{\partial m'}{\partial x}\right) + U_r\left(\frac{\partial m'}{\partial r}\right)\right] + \frac{\dot{m}}{m'} \end{aligned} \tag{11-18}$$

$$\nabla \times \vec{v}' = -\frac{1}{r}\left(\frac{\partial^2 \zeta'}{\partial x^2} + \frac{\partial^2 \zeta'}{\partial r^2}\right) = \left(\frac{\partial v'_r}{\partial x} - \frac{\partial v'_x}{\partial r}\right) \tag{11-19}$$

The foregoing relations may be interpreted in a simple way. The relative velocity components v'_x and v'_r are seen to depend on both a velocity potential function θ' and on a stream function ζ'. Only the velocity potential θ' is constrained by a continuity relation (11-18). The constraint on the stream function ζ' is implicated later, in connection with the equations of motion.

Note that once the basic flow $\vec{U}$ is known, the subsequent primary perturbation flows are fully describable by three scalar functions, namely, ψ', θ' and ζ'. In order to complete the analytical solution for these three functions, we must consider the application of the equations of motion to the primary perturbation, as discussed in a later section.

In the case of a uniform basic flow in a circular duct, the present relations do not necessarily simplify to any great extent because the perturbation flows $\vec{u}'$ and $\vec{v}'$ can be strongly non-uniform. About the only obvious simplification is that $U_r = 0$ in Eq. (11-18).

However, the degree of simplicity or complexity of the primary perturbation flow depends strongly on just how the particles are introduced into the main stream!

We have already seen that the introduction or elimination of particles in certain volumetric regions can be expressed mathematically by positive or negative values of the generation function $\dot{m}$. However, the particles are injected into the main stream with certain initial velocities which must also be specified in order to make the solution determinate. The convenient way to state this condition is in terms of the initial relative velocity vector, call it $\vec{v}'_o$, with components v'_{ox} and v'_{or}.

Of course, the introduction of particles can alternatively be regarded as distributed over certain specified surfaces or lines if that happens to be more convenient than generation over volume. In that case the character and units of the generation function $\dot{m}$ must be adjusted accordingly. For example, in the case of the uniform duct, the particles may be introduced over an inlet cross-section, say at $x = 0$, and collected at an exit cross-section, say at $x = \ell$. Then $\dot{m}$ refers to mass of particles introduced or removed per unit area per unit time. Because of the axial symmetry, $\dot{m}$ becomes a function of r only. At the inlet station this function, call it $\dot{m}_o(r)$, may be specified arbitrarily. The corresponding function at the exit station, $\dot{m}_\ell(x)$, if it is to represent complete collection of all of the particles, cannot in general be specified arbitrarily, but must be determined from the detailed solution. However, for the special case where the particles move in paths parallel to axis, then $\dot{m}_\ell(r) = -\dot{m}_o(r)$.

Specializing even further, consider the case where

$$\dot{m}_o(r) = cU_x(r) \tag{11-20}$$

where c is a constant. Suppose further that the particles are introduced in such a way that they have initially zero slip velocity, $\vec{v}_o' = 0$. It is at once evident for this strongly idealized case, without need of any further calculations, that the particles then simply continue to share the simple motion of the basic mean flow. The latter therefore remains essentially unperturbed, with $\vec{u}' = 0$. Moreover, it is easy to see that by reason of (11-20), the final particle density

$$m' = \text{constant} \tag{11-21}$$

over the entire region between inlet and outlet!

The above special case provides an attractive basis for futher development because of its relative simplicity. This is in line with our initial effort which on the one hand attempts to outline a procedure for a rather generalized analysis, while on the other it seeks to exploit the simplest available examples in order to help make the basic trends clear.

Of course, it must be recognized that in order to achieve the foregoing idealized state of uniform density, and zero primary perturbation, an energy input is required. This is the energy involved in injecting the particles at just the right velocity at each point so as to meet the no slip condition, and at just the right rate to meet the constant density condition.

Fortunately, however, this theoretical injection energy required is simply and precisely calculable.

An additional remark is required in connection with the case where the particles are formed by condensation of the carrier fluid itself, rather than being additional masses introduced from another source. In that case the generation of particles by condensation is accompanied by an equivalent dimunition of the carrier phase, and vice versa. Theoretically, the right side of Eqs (11-1) should be modified to reflect this variation. However, we assume that the total mass flow rate of the particles is always negligibly small compared with the mass flow rate of the carrier fluid. Hence the error involved in neglecting the above mentioned correction term in (11-1) is also negligible. Moreover, the use of (11-1) in the form given permits the employment of the stream function ψ'. This latter point is a considerable mathematical convenience, and more than justifies the method adopted.

12.0 TURBULENT DIFFUSION

In the present context, the term diffusion refers to the net tendency of the particles to migrate in the direction of decreasing density, that is, in the direction of the negative gradient $-\nabla m'$. The diffusion results principally from the mixing effect of the turbulence. Theoretically, there is also some diffusion arising from motions on the molecular level, but this contribution is entirely negligible compared with the turbulent diffusion, and need not be considered. Of course, a detailed solution of the actual turbulent fluctuations would establish the diffusion effects exactly, but this course is not practical for the reasons previously explained. However, the net mean effect of the diffusion can be represented in the customary manner by means of an appropriate diffusion coeficient D.

A brief and simplified analysis of the turbulent diffusion process is appropriate at this point. Refer to the sketch, Fig. 12-1.

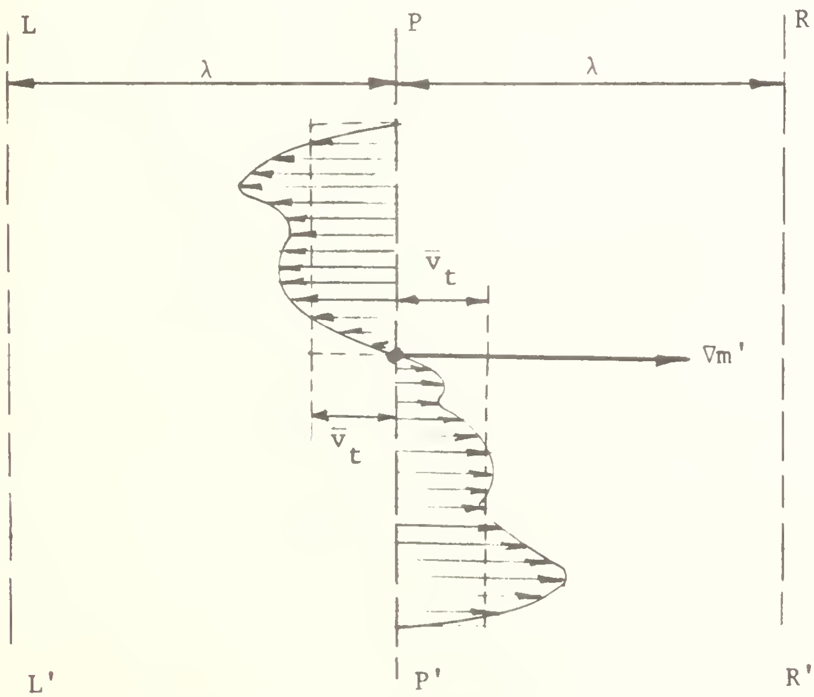

Figure 12-1 Schematic Diagram for Diffusion Analysis

Let PP' represent an arbitrary cross-section normal to the density gradient $\nabla m'$. At any instant of time there are flows in both directions across this plane. These are the flows associated with the turbulent eddies themselves and are in addition to the gross mean motion. Consider unit area in plane PP'. We consider that, on the average, flow occurs over half this area in the direction $+\nabla m'$, and over the other half in the direction $-\nabla m'$. Call the mean turbulent velocity fluctuation in either direction $\overline{v}_t$; note that we assume $\overline{v}_t$ to be the same in the two directions. On the average, fluid elements crossing plane PP' in the sense of $(+\nabla m')$ are coming from the vicinity of plane LL' at some distance λ from PP'. Similarly, fluid elements crossing plane PP' in the sense of $(-\nabla m')$ are coming from the vicinity of plane RR' at an equal distance λ on the other side of PP'. We assume in this over-simplified account that the particles are entrained by the carrier fluid and share its turbulent motion exactly. Actually there would be some slip between the particles and the carrier fluid, but we neglect this effect here. Consequently the dimensionless mass transport of particles in the two directions may now be represented in the form

$$\dot{m}_p^{(+)} = +\frac{1}{2}\left(m' - |\nabla m'|\lambda\right)\overline{v}_t$$
$$\dot{m}_p^{(-)} = -\frac{1}{2}\left(m' + |\nabla m'|\lambda\right)\overline{v}_t \qquad (12\text{-}1)$$

The net mass transport is therefore

$$\Delta\dot{m}_p = \dot{m}_p^{(-)} + \dot{m}_p^{(+)} = -(\lambda\overline{v}_t)\,\nabla m' \qquad (12\text{-}2)$$

The above net mass transport of particles in the direction of $(-\nabla m')$ may be regarded as equivalent to a mean effective mass flow rate of the particles in the direction $(-\nabla m')$ at some mean velocity, call it the "diffusion velocity" $\vec{w}'$. Therefore

$$\Delta \dot{m}_p = \vec{w}' \; m' = -(\lambda \bar{v}_t) \; \nabla m' \tag{12-3}$$

Using vector notation, we may solve this for the mean effective diffusion velocity vector, namely,

$$\vec{w}' = -(\lambda \bar{v}_t) \left(\frac{\nabla m}{m}\right)' \tag{12-4}$$

The factor of proportionality in (12-3) and (12-4) is the so-called diffusion coefficient, that is,

$$\lambda \bar{v}_t = D \tag{12-5}$$

Thus from a pragmatic viewpoint, the problem of allowing adequately for diffusion effects amounts to the problem of estimating an appropriate diffusion coefficient. Unfortunately, the diffusion coefficient D is in general a complicated function that can vary strongly over the flow field.

Eq. (8-5) shows that basically, D is the product of a "mixing velocity" $\bar{v}_t$ and a "mixing length" λ. Clearly $\bar{v}_t$ is associated with the local intensity of turbulence. It is apparent that $\bar{v}_t$ must be closely related to the turbulent energy. Thus

$$\bar{v}_t \sim \sqrt{2E} \;\; = \;\; [\bar{u}^2 + \bar{v}^2 + \bar{w}^2]^{1/2} \tag{12-6}$$

where u, v, w are the turbulent velocity fluctuations.

The mixing length λ is associated with the size of the turbulent eddies. This is a complicated matter and will not be discussed here.

An informative qualitative and quantitative analysis of this question may be found in reference (4).

Instead we shall here take advantage of the fact that there is a very close analogy and relation between the diffusion coefficient D and the eddy viscosity ε. Thus the Reynolds stresses may be analyzed in terms of momentum transport, the transport of momentum being proportional to the momentum gradient, with ε as the factor of proportionality. There is a close analogy here with the transport of mass. As we have seen, the mass transport is proportional to density gradient, with D as the factor of proportionality. Also the corresponding dimensional quantities $\hat{D}$ and $\hat{\varepsilon}$ have identical units. This very close physical analogy justifies us in postulating a simple relation between D and ε in the form

$$D = \gamma\varepsilon \tag{12-7}$$

where γ is a dimensionless number which should hardly differ much from unity. The best way to evaluate γ would be by experiment since there is no very straight forward and practical method to calculate it. Unfortunately, the available experimental information on this point seems sparse and unreliable. Probably the best recourse for the present is simply to assume $\gamma = 1$. The error involved in estimating diffusion effects in this way should not be excessive, and the resulting error in the overall solution should be well within the limits of accuracy which are appropriate for the present model, considering the other approximations and idealizations involved.

On this basis, Eq. (12-4) may be rewritten

$$\begin{aligned} \vec{w}' &= -\gamma\varepsilon \frac{\nabla m'}{m'} \\ \gamma &\simeq 1 \end{aligned} \tag{12-8}$$

Since ε becomes a known function over the field as soon as the basic flow has been analyzed in the manner explained earlier, Eq. (8-8) then suffices to fix the effective "diffusion velocity" w' of the particles at all points in the flow. In this way the mean effect of the turbulent diffusion is adequately simulated despite the fact that the detailed turbulent motion itself remains undefined.

There is another useful simplification that can be made in applying Eq. (12-8) to the primary and secondary perturbations. Denoting these by single and double primes, respectively, we may write

$$w' = -\gamma\varepsilon \left(\frac{\nabla m'}{m'}\right) \tag{12-9}$$

and

$$w'' = -\gamma\varepsilon \left(\frac{\nabla m''}{m' + m''}\right) \doteq -\gamma\varepsilon \left(\frac{\nabla m''}{m'}\right) \tag{12-10}$$

The corresponding scalar components are obvious.

Notice that the unprimed quantity $\gamma\varepsilon$ is used in these two relations. This amounts to assuming that the effect of the primary and secondary perturbations on the basic diffusion coefficient is negligible. Also notice that in (12-10) the quantity m' is used instead of $(m' + m'')$ in the denominator.

These minor modifications amount to linearizing assumptions which greatly simplify the subsequent analysis without appreciably impairing its accuracy.

Now consider the possible simplifications that can occur in applying the foregoing to the special case of a uniform circular duct. It has been shown in the previous section that it is theoretically possible in this case to inject the particles in just such a way that

$$m' = \text{constant} \tag{12-11}$$

If this in fact be done, it is then quite clear from the foregoing that all primary diffusion effects vanish, and

$$\vec{w}' = u' = 0 \tag{12-12}$$

13.0 VISCOUS FORCE ON A PARTICLE

We assume that the entrained particles are large compared with the mean free path of the carrier fluid. Consequently we may treat the fluid as a continuum and employ the no slip condition at the contact boundary between particle and fluid. The particle, while large compared with the mean free path, is still very small, small enough that surface tension effects suffice to maintain it in an essentially spherical form. Moreover, in our highly idealized model, all particles are taken to be identical in size, although in reality a continuous distribution of sizes is involved.

Under the foregoing idealized circumstances, Stoke's law of viscous drag applies. For the present application this may be written in dimensionless variables as

$$\vec{F}'_v = -6\pi\, \nu R_p (\vec{v}' - \vec{w}') \tag{13-1}$$

where the primes are used, for definiteness, to indicate the primary perturbation. The symbols are defined as follows

$\vec{F}'_v$ = dimensionless net viscous drag force on a single spherical particle.

ν = dimensionless kinenatic viscosity of carrier fluid.

R_p = dimensionless radius of particle.

$\vec{v}'$ = dimensionless total relative velocity of particle with respect to carrier fluid.

$\vec{w}'$ = dimensionless mean diffusion velocity

$(\vec{v}'-\vec{w}')$ = dimensionless net slip velocity of particle with respect to carrier fluid.

We now define the following related quantity

$\vec{f}'_v$ = dimensionless total viscous force exerted by the particles on the carrier fluid, per unit mass of carrier fluid. (The components of $\vec{f}'_v$ are f'_{vx} and f'_{vr}).

Then

$$\vec{f}'_v = -\frac{m'}{m_p}\vec{F}'_v = +6\pi\left(\frac{\nu R_p}{m_p}\right)m'(\vec{v}' - \vec{w}') \tag{13-2}$$

Eqs. (13-1) and (13-2) now fix the net viscous force acting, respectively, on the individual particle and on the carrier fluid for the primary perturbation. For the secondary perturbation, we simply replace the single primed variables ($\vec{f}'_v$, $\vec{v}'$, $\vec{w}'$, etc.) by corresponding double primed variables ($\vec{f}''_v$, $\vec{v}''$, $\vec{w}''$, etc.).

The forms of Eqs. (13-1) and (13-2) are not affected thereby. Recall, however, that in evaluating $\vec{w}'$ and $\vec{w}''$, the linearized relations (12-9) and (12-10), respectively, should be employed. When this is done, the results may be arranged in the form

$$\begin{aligned}\vec{f}'_v &= 6\pi\left(\frac{\nu R_p}{m_p}\right)(m'\vec{v}' + \gamma\varepsilon\nabla m') \\ \vec{f}''_v &= 6\pi\left(\frac{\nu R_p}{m_p}\right)(m'\vec{v}'' + \gamma\varepsilon\nabla m'')\end{aligned} \tag{13-3}$$

14.0 PARTICLE MOTION FOR PRIMARY PERTURBATION

In developing the perturbation equation for the particle motion, it is helpful to use vector methods. Note that for the primary perturbation, electrical forces are not yet in action. Consequently the equation of motion of a single particle may be written simply

$$\rho_p v_p \left\{(\vec{U} + \vec{v}' + \vec{v}')\cdot\nabla(\vec{U} + \vec{u}' + \vec{v})\right\} = -v_p \nabla(P + p') + \vec{F}'_v \quad (14\text{-}1)$$

where ρ_p, v_p, and $\vec{F}'_v$ represent the dimensionless density, volume, and viscous force pertaining to a single particle. Of course the particle mass is

$$m_p = \rho_p v_p \quad (14\text{-}2)$$

We divide (14-1) through by v_p and note from (13-2) that

$$\frac{\vec{F}'_v}{v_p} = -\frac{\rho_p}{m'}\vec{f}'_v \quad (14\text{-}3)$$

Therefore (14-1) becomes

$$\rho_p \left\{(\vec{U} + \vec{v}' + \vec{v}')\cdot\nabla(\vec{U} + \vec{u}' + \vec{v}')\right\} = -\nabla(P + p) - \frac{\rho_p}{m'} f'_v \quad (14\text{-}4)$$

On the other hand, for the carrier fluid itself, the corresponding equation of motion is merely

$$\left\{(\vec{U} + \vec{u}')\cdot\nabla(\vec{U} + \vec{u}')\right\} = -\nabla(P + p') + \vec{f}'_v \quad (14\text{-}5)$$

The pressure term is undesirable in (14-4). We can eliminate it by subtracting (14-5) from (14-4). This amounts to representing the particle motion itself as a kind of a perturbation with respect to the motion of

the carrier fluid. The result is

$$\rho_p\left\{(\vec{U} + \vec{u}' + \vec{v}')\cdot\nabla(\vec{U} + \vec{u}' + \vec{v}')\right\} - \left\{(\vec{U} + \vec{u}')\cdot\nabla(\vec{U} + \vec{u}')\right\}$$

$$= -\left(1 + \frac{\rho_p}{m'}\right) \vec{f}'_v \doteq -\frac{\rho_p}{m'}\vec{f}'_v \tag{14-6}$$

The left side can now be expanded and simplified. In this process we can neglect quantities which are quadratic in the perturbations. We also take advantage of the fact that the perturbations u' and v' are small compared with the basic velocity U and may therefore be neglected where appropriate. The resulting linearized equation reduces to

$$\vec{U}\cdot\nabla\vec{v}' + \vec{v}'\cdot\nabla\vec{U} = -\frac{\vec{f}'}{m'} - \left(1 - \frac{1}{\rho_p}\right)\vec{U}\cdot\nabla\vec{U} \tag{14-7}$$

Also recall from (13-3) that the viscous force is

$$\vec{f}'_v = 6\pi\left(\frac{\nu R_p}{m_p}\right)(m'\vec{v}' + \gamma\varepsilon\nabla m') \tag{14-8}$$

Upon substituting (14-8) into (14-7) and rearranging, we obtain

$$\begin{aligned}\vec{U}\cdot\nabla\vec{v}' + \vec{v}'\cdot\nabla\vec{U} + 6\pi\left(\frac{\nu R_p}{m_p}\right)\vec{v}' &= -6\pi\left(\frac{\nu R_p}{m_p}\right)\gamma\varepsilon\left(\frac{\nabla m'}{m'}\right) \\ &\quad -\left(1 - \frac{1}{\rho_p}\right)\vec{U}\cdot\nabla\vec{U}\end{aligned} \tag{14-9}$$

Recall also the continuity relation (11-12) which we repeat here for convenience, namely,

$$\nabla\cdot\vec{v}' = -\vec{U}\cdot\frac{\nabla m'}{m'} + \left(\frac{\dot{m}}{m'}\right) \tag{14-10}$$

Eqs. (14-9) and (14-10) are now the two basic equations which jointly govern the relative density distribution m' of the particles, and the relative velocity $\vec{v}'$ of the particles with respect to the mean flow.

The functions $\vec{U}$ and $\gamma\varepsilon$ which occur in Eqs. (14-9) and (14-10) are at this stage known from the prior analysis of the basic flow field. The last term on the right of Eq. (14-9) represents an inertia effect associated with the fact that the particle density differs from that of the carrier fluid. The first term on the right of (14-9) represents the effect of turbulent diffusion.

Notice the significant fole in this governing equation of the dimensionless parameter

$$6\pi\left(\frac{\nu R_p}{m_p}\right) = 6\pi \frac{\left(\frac{\hat{\nu}}{\hat{V}\hat{\ell}}\right)\left(\frac{\hat{R}_p}{\hat{\ell}}\right)}{\left(\frac{\frac{4}{3}\pi \hat{R}_p^3 \hat{\rho}_p}{\hat{\rho}\,\hat{\ell}^3}\right)} = \frac{9}{2}\left(\frac{\hat{\rho}}{\hat{\rho}_p}\right)\left(\frac{\hat{\ell}}{\hat{R}_p}\right)^2\left(\frac{\hat{\nu}}{\hat{V}\hat{\ell}}\right) \tag{14-11}$$

The last factor on the right of Eq. (14-11) is seen to be the inverse of the ordinary Reynolds number of the overall flow. From its role in Eq. (14-9), the expression (14-11) is seen to represent, in a generalized way, the ratio of the viscous to the inertia forces acting on a fluid particle. Clearly, this is one of the fundamental parameters of the problem. The effect of systematically varying this parameter should be studied in the subsequent calculation phase of this research.

The equations (14-9) and (14-10), along with the appropriate boundary conditions on m' and $\vec{v}'$, are the necessary and sufficient conditions for determining the detailed solution for m' and $\vec{v}'$. The boundary conditions on $\vec{v}'$ are simply stated. Firstly, $\vec{v}'$ has some arbitrary but definite distribution, say $\vec{v}'_o$, in the region where the particles are initially injected.

Secondly, $\vec{v}'$ satisfies a no-slip condition at all solid boundaries. In regard to the boundary conditions on m', it is necessary only to specify some arbitrary but definite injection rate function $\dot{m}$ over the region where particles are injected.

In working out the detailed numerical solution for v' and m' from Eqs (14-9) and (14-10), it may be convenient to express v' in terms of its two potential functions θ' and ζ' according to Eq. (11-14), that is,

$$\vec{v}' = \nabla\theta + \frac{1}{r}\nabla\zeta' \times \vec{e}_\theta \qquad (14\text{-}12)$$

This relation may be substituted into Eqs. (14-9) and (14-10) if desired. In that case, the problem of the particle motion reduces to the solution of the three scalar functions θ', ζ', and m'. Since (14-9) is a vector equation, it amounts to two scalar equations. Hence Eqs. (14-9) and (14-10) then reduce to three simultaneous partial differential equations in the three unknowns θ', ζ', and m'.

This analysis shows that while the generalized solution for the primary perturbation is fairly complex, it does lie within the present state of the art of digital computation.

Turning now to the special case of the uniform circular duct, we again consider the idealized example in which the injection process is so regulated that the particles are introduced with zero slip velocity, and at a rate which maintains the relative density m' everywhere constant. Of course, in this limiting case, the entire primary perturbation process simply vanishes. We obtain the simple results

$$\begin{aligned} \vec{u}' &= \vec{v}' = 0 \\ \vec{f}'_v &= 0 \qquad (14\text{-}13) \\ m' &= \text{constant} \end{aligned}$$

Nevertheless, this result is not entirely trivial. In fact it does supply a particularly convenient and therefore useful basis for analysis of the secondary perturbation as considered in a later section.

15.0 EQUATIONS OF MOTION OF CARRIER FLUID FOR PRIMARY PERTURBATION

Consider the vorticity transport equation (10-3) of the carrier fluid. In vector terms this may be written

$$\vec{U}\cdot\nabla\vec{\Omega} = \nabla x\vec{f} \tag{15-1}$$

where

$$\vec{f} = \nabla\cdot(\varepsilon\vec{\Omega}) \tag{15-2}$$

The primary perturbation of these equations gives

$$(U' + \vec{u}')\cdot\nabla(\vec{\Omega} + \vec{\omega}') = \nabla x(\vec{f} + \vec{f}') + \nabla x\vec{f}_v \tag{15-3}$$

where

$$(\vec{f} + \vec{f}') = \nabla\cdot[\varepsilon(\vec{U} + \vec{u}')] \tag{15-4}$$

and

$$\vec{f}'_v = 6\pi\left(\frac{\nu R_p}{m_p}\right) (m'\vec{v}' + \gamma\varepsilon\nabla m') \tag{15-5}$$

Next we subtract the original equations from the perturbed equations. Moreover, the perturbations are small and quantities quadratic in the perturbations may therefore be neglected. Note also from (15-3) that the eddy viscosity function ε is assumed to be unaffected by the perturbation. Consequently we obtain finally the required perturbation equation, namely,

$$\vec{U}\cdot\nabla\vec{\omega} + \vec{u}'\cdot\nabla\vec{\Omega} = \nabla x\vec{f} + \nabla x\vec{f}_v \tag{15-6}$$

where

$$\vec{f}' = \nabla\cdot(\varepsilon\nabla\vec{u}') \tag{15-7}$$

and

$$\vec{f}_v = 6\pi\left(\frac{\nu R_p}{m_p}\right)(m'\vec{v}' + \gamma\varepsilon\nabla m') \tag{15-8}$$

These last three expressions may be combined into a single equation, namely,

$$\vec{U}\cdot\nabla(\nabla x\vec{u}') + \vec{u}'\cdot\nabla(\nabla x\vec{U}) = \nabla x\left\{\nabla\cdot(\varepsilon\vec{u}') + 6\pi\left(\frac{\nu R_p}{m_p}\right)(m'\vec{v}' + \gamma\varepsilon\nabla m')\right\} \tag{15-9}$$

Eq. (15-9) is now the basic relation which governs the perturbation $\vec{u}'$. Taken together, the three relations (14-9), (14-10), and (15-9), when combined with the appropriate boundary conditions, provide the necessary and sufficient equations which determine the three primary perturbation quantities $\vec{v}'$, m' and $\vec{u}'$.

Of course, the foregoing vector relations all have their scalar counterparts. In actual computations it is the scalar form of the equations which must be used. However, the general method of writing the scalar forms of the various vector relations has already been well illustrated in earlier sections of this report. Hence we shall refrain from writing out at this point the detailed scalar equivalents of the above equations, and proceed instead to the next phase of the analysis.

Once the functions $\vec{v}'$, m', and $\vec{u}'$ have been found by solution of Eqs. (14-9), (14-10), and (15-9) it is next required to determine the corresponding pressure perturbation p'. This can be found by perturbation of the basic pressure equation (10-5). The procedure is analogous to

that employed in deriving (15-9) above. Once again, quantities quadratic in the perturbations are neglected and ε is regarded as invariant. The final result is simply

$$\nabla^2 p' = -\nabla\cdot\left\{\vec{U}\cdot\nabla\vec{u}' + \vec{u}'\cdot\nabla\vec{U} + \nabla\cdot(\varepsilon\vec{u}')\right\} \tag{15-10}$$

16.0 THE SECONDARY PERTURBATION: ELECTRICAL EFFECTS

The secondary perturbation comprises all those effects which are associated with the addition of electrical charges and electrical forces to the model.

It is assumed that at the time and place each particle is injected into the flow field, it receives an electrical charge of amount q_p. In keeping with our previous assumption that all particles are of identical size and mass, we assume also that all particles carry identical charge q_p.

At some point downstream, the particle is assumed to reach a collector electrode where it gives up its electrical charge; the particle itself may or may not be eliminated also at this point.

A strong electrical potential difference is impressed across the flow field. Normally all points where charged particles enter the flow will be at some fixed potential, say $\hat{\phi}_o$, and all points on the collector electrode will be at some other fixed potential, say $\hat{\phi}_1$. In the flow region between these two stations, the electrical potential $\hat{\phi}''$ and its dimensionless counterpart ϕ'' will vary spatially in some manner which remains to be determined.

In passing through the field, each charged particle is subject to an electrical force $\vec{F}''_E$. This force, and hence the electrical field

which produces it, may either do work on the particle, or have work done upon it by the particle, depending on the algebraic signs of the electrical charge q_p, and of the overall potential difference $\Delta\hat{\phi} = (\hat{\phi}_1 - \hat{\phi}_o)$.

The movement of the electrical charges as they ride along with their respective particles constitutes an electrical current. However, the particle density, charge density, and current flux are all at extremely low levels. Thus magnetic effects and all effects associated with high current flux are negligible. On the other hand, electrical potential differences and potenial gradients are very high. In fact, ideally it is desirable to make them as high as possible. There is, of course, a limit to the potential gradient which can be achieved without causing electrical breakdown. This limit depends on the particular carrier fluid used, on its pressure, on the turbulence intensity, and so forth.

Under these conditions, Maxwell's basic equations of the electrical field simplify drastically. The dimensionless charge density ρ_e'' can be expressed in terms of the particle relative mass density (m' + m'') through the simple proportionality

$$\rho_e' = \left(\frac{q_p}{m_p}\right) (m' + m'') \doteq \left(\frac{q_p}{m_p}\right) m' \tag{16-1}$$

Notice that the secondary perturbation density m'' has been neglected in comparison with the primary perturbation density m'. Particle mass is injected only in connection with the primary perturbation; there is no additional mass injection associated with the secondary perturbation. Hence the secondary mass distribution m'' represents the small changes in the basic distribution m' produced by the electrical forces alone. But since the electrical forces are relatively weak, m'' is in general small compared with m'. Moreover, neglecting m'' in comparison with m' in effect linearizes and simplifies the solution procedure.

Even if the calculations be extended into the range where m'' is no longer negligible compared with m', we may, for a first approximation, still neglect m''. In that case, however, subsequent iterations will be required to bring in the true effect of m''.

In view of (16-1), one of the fundamental Maxwell relations requires that

$$\nabla^2 \Phi'' = -\frac{\rho_e''}{\varepsilon_o} = -\left(\frac{q_p}{\varepsilon_o m_p}\right) m' \tag{16-2}$$

where ε_o is the effective dielectric constant of the flow field.

Eq. (16-2) is an equation of the Poisson type. Because of the linearizing assumption introduced above, the right side of (16-2) is now a known function. Hence Eq. (16-2) may be integrated at once, subject to the specified boundary conditions on Φ''. This therefore fixes the detailed distribution of the dimensionless electric potential Φ'' over the entire field.

The electrical force on each single particle is then also a known function, and is given by the expression

$$\vec{F}_E'' = -q_p \nabla \Phi'' \tag{16-3}$$

It is convenient to change the reference and express the electrical force acting on the particles in terms of force per unit mass of carrier fluid. This gives

$$\vec{f}_E'' = \frac{(m' + m'')\,\vec{F}_E''}{m_p} \doteq \frac{m'\vec{F}_E''}{m_p} = -\frac{m' q_p}{m_p} \nabla \phi'' \tag{16-4}$$

For steady flow conditions, conservation of charge requires that

$$\nabla \cdot \vec{J} = \nabla \cdot \left\{ \left(\frac{m_p}{q_p}\right) (m' + m'')(\vec{U} + \vec{u}' + \vec{u}'' + \vec{v}' + \vec{v}'') \right\} = 0 \tag{16-5}$$

However, since $\frac{q_p}{m_p}$ is a constant it cancels from this relation. What remains is seen to be equivalent to an ordinary continuity equation, so that (12-5) does not really specify any additional constraint.

It follows from the foregoing that the essential electrical effects pertinent to this problem are expressed by just two basic equations, namely Eqs. (16-2) and (16-4).

The particle motion for the secondary perturbation can be analyzed in much the same way as for the primary perturbation. The essential difference now is the addition of the electrical forces.

Writing the equation of motion for a single particle, and dividing through by the volume v_p of the particle, we obtain

$$\rho_p\left\{(\vec{U} + \vec{u}' + \vec{u}'' + \vec{v}' + \vec{v}'')\cdot\nabla(\vec{U} + \vec{u}' + \vec{u}'' + \vec{v}' + \vec{v}'')\right\}$$

$$= -\nabla(P + p' + p'') - \rho_p\left(\frac{\vec{f}'_v}{m'} + \frac{\vec{f}''}{m' + m''}\right) - \rho_p \frac{q_p}{m_p} \nabla\phi'' \tag{16-6}$$

The left side of this equation represents inertia effects while the terms on the right represent the pressure, viscous and electrical forces, respectively. Once again we shall neglect m'' in comparison with m' in the above equation.

For the carrier fluid itself, the equation of motion becomes

$$\left\{(\vec{U} + \vec{u}' + \vec{u}'')\cdot\nabla(\vec{U} + \vec{u}' + \vec{u}'')\right\}$$

$$= -\nabla(P + p' + p'') + (\overline{f}'_v + \overline{f}''_v) \tag{16-7}$$

The unknown pressures can be eliminated by subtracting (16-7) from (16-6). Then dividing through by ρ_p, and rearranging we obtain

$$
\begin{aligned}
&\left(1 - \frac{1}{\rho_p}\right) \left\{(\vec{U} + \vec{u}' + \vec{u}'')\cdot\nabla(\vec{U} + \vec{u}' + \vec{u}'')\right\} \\
&\quad + (\vec{U} + \vec{u}' + \vec{u}'')\cdot\nabla(\vec{v}' + \vec{v}'') + (\vec{v}' + \vec{v}'')\cdot\nabla(\vec{U} + \vec{u}' + \vec{u}'') \\
&\quad + (\vec{v}' + \vec{v}'')\cdot\nabla(\vec{v}' + \vec{v}'')\Big\} \\
&\quad = -\left(\frac{1}{\rho_p} + \frac{1}{m'}\right)\left(\vec{f}'_v + \vec{f}''_v\right) - \left(\frac{q_p}{m_p}\nabla\Phi''\right)
\end{aligned}
\tag{16-8}
$$

On the other hand, in the absence of any electrical forces this relation reduces to the form associated with the primary perturbation alone, namely,

$$
\begin{aligned}
&\left(1 - \frac{1}{\rho_p}\right) \left\{(\vec{U} + \vec{u}')\cdot\nabla(\vec{U} + \vec{u}')\right\} \\
&\quad + \left\{(\vec{U} + \vec{u}')\cdot\nabla\vec{v}' + \vec{v}\cdot\nabla(\vec{U} + \vec{u}') + \vec{v}'\cdot\nabla\vec{v}'\right\} \\
&\quad = -\left(\frac{1}{\rho_p} + \frac{1}{m'}\right)\vec{f}'_v
\end{aligned}
\tag{16-9}
$$

Upon subtracting (16-9) from (16-8) the following relation is obtained for the secondary perturbation itself, namely,

$$
\begin{aligned}
&\left(1 - \frac{1}{\rho_p}\right)\left\{(\vec{U} + \vec{u}')\cdot\nabla\vec{u}'' + \vec{u}''\cdot\nabla(\vec{U} + \vec{u}') + \vec{u}''\cdot\nabla\vec{u}''\right\} \\
&\quad + \Big\{(\vec{U} + \vec{u}' + \vec{u}'')\cdot\nabla\vec{v}'' + \vec{u}''\cdot\nabla(\vec{v}' + \vec{v}'') + \vec{v}''\cdot\nabla(\vec{U} + \vec{u}' + \vec{u}'') \\
&\quad + (\vec{v}' + \vec{v}'')\cdot\nabla\vec{u}'' + \vec{v}'\cdot\nabla\vec{v}'' + \vec{v}''\cdot\nabla\vec{v}' + \vec{v}''\cdot\nabla\vec{v}''\Big\} \\
&\quad = -\left(\frac{1}{\rho_p} + \frac{1}{m'}\right) \vec{f}''_v - \left(\frac{q_p}{m_p}\right)\nabla\Phi''
\end{aligned}
\tag{16-10}
$$

This relation can be linearized and simplified in the usual way. All perturbation quantities $\vec{u}'$, $\vec{u}''$, $\vec{v}'$, $\vec{v}''$ are negligible in comparison with $\vec{U}$ itself. Also $\frac{1}{\rho_p}$ is negligible in comparison with $\frac{1}{m'}$. Upon rearranging, we obtain the following result.

$$\vec{U}\cdot\nabla\vec{v}'' + \vec{v}''\cdot\nabla\vec{U} = -\frac{\vec{f}_v''}{m'} - \left(1 - \frac{1}{\rho_p}\right)(\vec{U}\cdot\nabla\vec{u}'' + \vec{u}''\cdot\nabla\vec{U}) - \left(\frac{q_p}{m_p}\right)\nabla\Phi'' \tag{16-11}$$

Recall from (9-3) that the viscous force is given by

$$\frac{\vec{f}_v''}{m'} = 6\pi\left(\frac{\nu R_p}{m_p}\right)\left(\vec{v}'' + \gamma\varepsilon\,\frac{\nabla m''}{m'}\right) \tag{16-12}$$

Substituting this expression into (16-11) and rearranging gives, finally,

$$\vec{U}\cdot\nabla\vec{v}'' + \vec{v}''\cdot\nabla\vec{U} + 6\pi\left(\frac{\nu R_p}{m_p}\right)v'' = -6\pi\left(\frac{\nu R_p}{m_p}\right)\gamma\varepsilon\left(\frac{\nabla m''}{m'}\right) - \left(1 - \frac{1}{\rho_p}\right)(\vec{U}\cdot\nabla\vec{u}'' + \vec{u}''\cdot\nabla\vec{U}) - \left(\frac{q_p}{m_p}\right)\nabla\Phi'' \tag{16-13}$$

This is the basic relation governing the secondary perturbation velocity v''. Of course, $\vec{v}''$ must also conform to the requirements of continuity.

For the secondary perturbation, the continuity equation becomes

$$\nabla\cdot[(m' + m'')(\vec{U} + \vec{u}' + \vec{u}'' + \vec{v}' + \vec{v}'')] = 0 \tag{16-14}$$

Expanding this, and noting that

$$\nabla\cdot U = \nabla\cdot u' = \nabla\cdot u'' = 0 \tag{16-15}$$

we obtain

$$(m' + m'')(\nabla\cdot\vec{v}' + \nabla\cdot v'') + (\nabla m' + \nabla m'')\cdot(\vec{U} + u' + \vec{u}'' + v' + \vec{v}'') = 0 \tag{16-16}$$

We now make the linearizing assumptions that have been used throughout, namely, that $\vec{u}'$, $\vec{u}''$, $\vec{v}'$, $\vec{v}''$ are negligible compared with U, and m'' is negligible compared with m'. Then rearranging, we find that

$$\nabla\cdot\vec{v}'' = -\nabla\cdot\vec{v}' - \vec{U}\cdot \frac{\nabla m'}{m'} \tag{16-17}$$

Recall, however, that the continuity equation (11-12) for the primary perturbation was

$$\nabla\cdot\vec{v}' = -\vec{U} \cdot \frac{\nabla m'}{m'} + \frac{\dot{m}}{m'} \tag{16-18}$$

Upon substituting (16-18) into (16-17) we obtain finally the result

$$\nabla\cdot\vec{v}'' = - \frac{\dot{m}}{m'} \tag{16-18}$$

Outside the limited region where particles are actually being injected, the right side vanished hence (16-19) reduces to the simple statement that

$$\nabla\cdot\vec{v}'' = 0 \tag{16-20}$$

It is clear from Eq. (16-20) that $\vec{v}''$ can be expressed in terms of a stream function ζ'' such that

$$v'' = \frac{1}{r}\nabla\zeta'' \times \vec{e}_\theta \tag{16-21}$$

Recall that, in contrast with this, the primary perturbation v' requires two potential functions for its definition, that is, that

$$\vec{v}' = \nabla\theta' + \frac{1}{r}\nabla\zeta' \times \vec{e}_\theta \tag{16-22}$$

It can be seen that Eqs. (16-13) and (16-19) or (16-20) jointly govern the variables v'' and m''. Moreover, it can be seen that Eq. (16-13) contains not only the unknowns $\vec{v}''$ and m'', but also $\vec{u}''$. Hence a third equation is needed to define a determinate solution.

The required relation is obtained by superimposing a secondary perturbation on the vorticity transport equation (10-3). The procedure is exactly analogous to that involved in Eqs. (15-1) through (15-9). Consequently we can immediately write the required result by direct analogy with (15-9). This gives

$$\vec{U}\cdot\nabla(\nabla\times\vec{u}'') + \vec{u}''\cdot\nabla(\nabla\times\vec{U}) = \nabla\times\left\{\nabla\cdot(\varepsilon\vec{u}'') + 6\pi\left(\frac{\nu R_p}{m_p}\right)(m'\vec{v}'' + \gamma\varepsilon\nabla m'')\right\} \tag{16-23}$$

Thus the solution for the three functions $\vec{v}''$, m'', and $\vec{u}''$ which fully define the secondary perturbation is governed by three simultaneous partial differential equations, namely, Eqs. (16-13), (16-19), and (16-23).

Once the above functions are known, the secondary perturbation pressure p'' can be found from a perturbation equation analogous to Eq. (15-10). This is

$$\nabla^2 p'' = -\nabla\cdot\left\{\vec{U}\cdot\nabla\vec{u}'' + \vec{u}''\cdot\nabla\vec{U} + \nabla\cdot(\varepsilon u'')\right\} \tag{16-24}$$

Now let us revert to the specialized case which has served as our example throughout this discussion, namely, the uniform circular duct. Again we confine attention to the theoretical situation where injection is so regulated as to produce $\vec{u}' = \vec{v}' = 0$ and constant relative particle density m' over the field. In that case, Eq. (12-2) becomes

$$\nabla^2\phi'' + \frac{\partial^2\phi''}{\partial x^2} + \frac{\partial^2\phi''}{\partial r^2} + \frac{1}{r}\frac{\partial\phi''}{\partial r} = -\left(\frac{q_p}{\varepsilon_o m_p}\right)m' = \text{constant} \tag{16-25}$$

The boundary conditions on dimensionless potential ϕ'' can be taken as

$$\phi'' = \phi_o = 0 \qquad \text{at } x = 0$$

and

$$\phi'' = \phi_1 = \pm 1 \qquad \text{at } x = \ell \tag{16-26}$$

For definiteness in this example, we arbitrarily choose the negative sign in this last expression. This choice corresponds to EHD power generation using negatively charged particles.

The solution of (16-25) subject to the boundary conditions (16-26) gives the dimensionless electric potential in the form

$$\phi'' = -\left\{\frac{x}{\ell} - \left(\frac{q_p m'}{m_p}\frac{\ell^2}{2}\right)\frac{x}{\ell}\left(1 - \frac{x}{\ell}\right)\right\} \tag{16-27}$$

The corresponding dimensionless potential gradient is

$$\nabla\phi'' = -\left\{1 - \left(\frac{q_p m'}{m_p}\frac{\ell^2}{2}\right)\left(1 - 2\frac{x}{\ell}\right)\right\}\vec{e}_x \tag{16-28}$$

Consequently the net electrical force per unit mass of carrier fluid becomes

$$\vec{f}''_F = -\frac{m'q_p}{m_p} = +\left(\frac{m'q_p}{m_p}\right)\left\{1 - \left(\frac{q_p m'}{m_p}\frac{\ell^2}{2}\right)\left(1 - 2\frac{x}{\ell}\right)\right\}\vec{e}_x \tag{16-29}$$

Notice that the force on the particles is purely axial and that since q_p is assumed negative in this instance, the electrical forces act in a sense such as to resist the motion. Consequently, the fluid in this case does work against the resistance of the electrical field, thus converting mechanical energy into electrical energy.

Continuation of the solution by purely analytical methods beyond this point becomes unduly cumbersome, even for the idealized example considered here.

However, continuation by numerical techniques lies within the capabilities of present computer technology. In this way it is now possible to investigate systematically a wide range of operating conditions and to determine corresponding performance parameters such as efficiency

and specific output. Comprehensive calculations of this kind should yield valuable information concerning optimum design values of the parameters and corresponding performance possibilities.

17.0 SUMMARY OF PRINCIPAL EQUATIONS

In this section we summarize the key equations only, in vector form for conciseness. These relations have already been discussed in detail in the report.

Mean Flow

$$U\cdot\nabla(\nabla \times \vec{U}) = \nabla\times[\nabla\cdot(\varepsilon\nabla U)] \qquad \text{Vorticity Transport} \qquad (17\text{-}1)$$

$$\varepsilon = \varepsilon(\grave{U}) \qquad \text{Eddy Viscosity Hypothesis} \qquad (17\text{-}2)$$

Primary Perturbation

$$\vec{U}\cdot\nabla\vec{v}' + \vec{v}'\cdot\nabla\vec{U} + 6\pi\left(\frac{\nu R_p}{m_p}\right)\vec{v}' =$$

$$\text{Particle Motion} \qquad (17\text{-}3)$$

$$-6\pi\left(\frac{\nu R_p}{m_p}\right)\gamma\varepsilon\left(\frac{\nabla m'}{m'}\right) - \left(1 - \frac{1}{\rho_p}\right)\vec{U}\cdot\nabla\vec{U}$$

$$\nabla\cdot\vec{v}' = -\vec{U}\cdot\frac{\nabla m'}{m'} + \frac{\dot{m}}{m'} \qquad \text{Conservation of Mass} \qquad (17\text{-}4)$$

$$\vec{U}\cdot\nabla(\nabla\times\vec{u}') + (\nabla\times\vec{u}')\cdot\nabla\vec{U} =$$

$$\text{Vorticity Transport} \qquad (17\text{-}5)$$

$$\nabla\times\left\{\nabla\cdot(\varepsilon\nabla u') + 6\pi\left(\frac{\nu R_p}{m_p}\right)(m'v + \gamma\varepsilon\nabla m')\right\}$$

Secondary Perturbation

$$\nabla^2\Phi'' = -\left(\frac{q_p}{\varepsilon_o m_p}\right)m' \qquad \text{Electric Potential} \qquad (17\text{-}6)$$

$$\vec{U}\cdot\nabla\vec{v}'' + \vec{v}''\cdot\nabla\vec{U} + 6\pi\left(\frac{\nu R_p}{m_p}\right)\vec{v}'' =$$

Particle Motion (17-7)

$$-6\pi\left(\frac{\nu R_p}{m_p}\right)\gamma\varepsilon\,\frac{\nabla m''}{m'} - 1 - \frac{1}{\rho_p}\left(\vec{U}\cdot\nabla\vec{u}'' + \vec{u}''\cdot\nabla\vec{U}\right) - \frac{q_p}{m_p}\nabla\Phi''$$

$$\nabla\cdot\vec{v}'' = -\frac{\dot{m}}{m'}$$

Conservation of Mass (17-8)

$$\vec{U}\cdot\nabla(\nabla x\vec{u}'') + (\nabla x\vec{u}'')\cdot\nabla\ (\nabla x\vec{U}) =$$

Vorticity Transport (17-9)

$$\nabla x\left\{\nabla\cdot(\varepsilon\ \vec{u}'') + 6\pi\left(\frac{\nu R_p}{m_p}\right)(m'\vec{v}'' + \gamma\varepsilon\nabla m'')\right\}$$

The above set of nine fundamental relations, plus the appropriate boundary conditions, constitute the necessary and sufficient equations for determining the mean motion of carrier fluid and charged particles at every point in the field. This mathematical model includes mass and momentum transport effects caused by turbulence.

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13. ABSTRACT

A mathematical model of the basic electro-hydrodynamic (EHD) process is developed and described. The carrier fluid is treated as incompressible and turbulent. The injected particles are treated as uniform in mass and electrical charge.. The analysis is broken down into three phases, namely, the basic flow, the perturbation due to injection of mass, and the perturbation due to introduction of electrical charge. This method greatly simplifies and improves the analysis. A final system of nine basic partial differential equations is obtained. These equations, along with the appropriate boundary conditions, fix the fluid and particle velocities and particle density at all points in the field. The basic equations are developed in a fully non-dimensional form.

The mathematical model here presented is unique in its analytical approach and in its treatment of turbulence effects. Through computer simulation, it offers new possibilities for the study and development of EHD power generation systems.

The analytical model has been developed to the point where it is ready for computer programming. Such a program would be useful for estimating optimum design parameters and performance possibilities for a wide variety of axi-symmetric configurations and a wide range of operating conditions.

Unfortunately the work has now been halted by shortage of funds. Because of the value of this research, it is recommended that this project be resumed and continued into the next stage, which is the stage of actual programming and computation.

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